THE JOHNS HOPKINS UNIVERSITY
Department of Mechanics
Baltimore, Maryland

January, 1963

THE USE OF THE GOVERNING EQUATIONS IN DIMENSIONAL ANALYSIS

By Robert R. Long

Technical Report No. 13 (ONR Series)
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Abstract

This paper develops the idea that the most effective use of dimensional analysis in fluid mechanics is to combine it with the known mathematical form of the governing equations. It is shown that additional information can frequently be obtained in this way, especially when physical approximations can be reasonably made.

THE USE OF THE GOVERNING EQUATIONS IN DIMENSIONAL ANALYSIS

1. Introduction

One of the nice things about the science of fluid mechanics is that we have a very good idea of the mathematical form of the equations that govern our problems even in very complicated cases, for example, problems of heat convection. The task of solving these equations is a severe one, of course, and as a result we do as other scientists in other fields and resort to approximation and experiment. But whichever we do, we find that a reasoning process, called dimensional analysis, is useful not only to guide us in our tasks but, frequently, to give us directly answers to some of the questions that face us. For example, it is convenient for experimental purposes to know that the drag on a body in a stream of fluid of speed u has the form

$$F = \rho u^2 f(Re)$$

where ρ is the density and Re is the Reynolds number. It is also informative to have this product of dimensional reasoning. For example, it tells us that in two different fluids of the same kinematic viscosity, the drag is proportional to the density.

But the theme of this paper is not that dimensional analysis is convenient and informative both for the experimenter and the theoretician, although this is both true and important. Rather, the theme is that when we know the mathematical form of our governing equations, we should exploit this fact not only by searching for solutions to these equations, but also by incorporating as much as possible of this knowledge of our equations into the reasoning process of dimensional analysis.

2. Generalized dimensional analysis

I illustrate these remarks by the problem from elementary mechanics of finding the range of a projectile (Fig. 1).

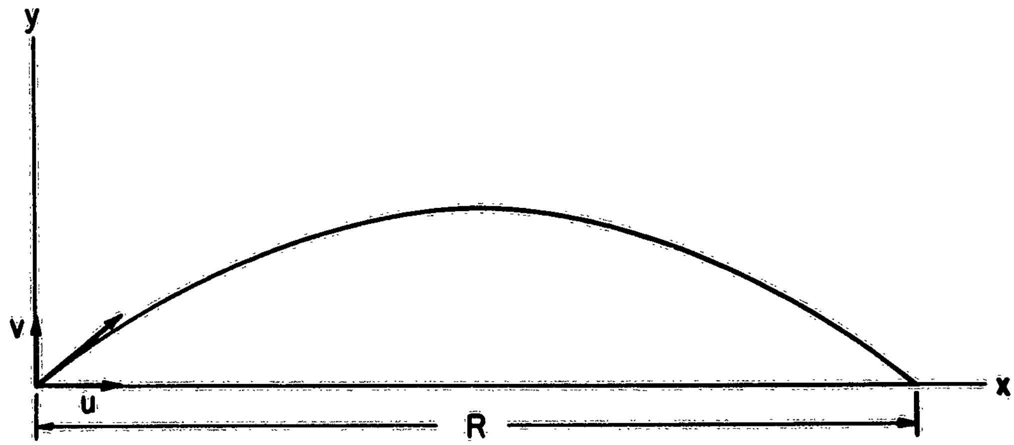

Fig. 1

The problem is completely defined by the set of equations and conditions.

$$\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = -g \qquad \begin{aligned} x(0) &= 0 \\ y(0) &= 0 \\ \dot{x}(0) &= u \\ \dot{y}(0) &= v \end{aligned} \qquad \text{at } y = 0,\ x = R \tag{1}$$

Obviously the range R depends only on u, v, g, and, using the physical dimensions of these 4 quantities, the π-theorem (Bridgman, 1931) yields

$$R = \frac{u^2}{g} f\left(\frac{v}{u}\right) \tag{2}$$

On the other hand, let us allow all the quantities that enter the system (1) to have arbitrary dimensions. For example, let the dimension of x be A, the dimension of y be B, etc. Symbolically we write,

$$[x] = A, \quad [t] = C, \quad [v] = E, \quad [R] = G$$

$$[y] = B, \quad [u] = D, \quad [g] = H$$

Now require that all equations in (1) be dimensionally homogeneous, for example $[x] = [R]$, etc. The resulting relationships among A, B, etc., lead to

$$[x] = A, \quad [t] = C, \quad [v] = \frac{B}{C}, \quad [R] = A$$

$$[y] = B, \quad [u] = \frac{A}{C}, \quad [g] = \frac{B}{C^2},$$

Hence the four quantities R, u, v, g have three dimensions, A, B, C, instead of the two physical dimensions L and T, and the π-theorem yields

$$\frac{Rg}{uv} = K \tag{3}$$

where K is a nondimensional constant. Equation (3) is much more informative, of course, than equation (2). In fact (3) shows that the function F, which is arbitrary in equation (2), must have the form

$$f\left(\frac{v}{u}\right) = K\frac{v}{u} \tag{4}$$

This example illustrates the power of generalized dimensional analysis.* The equations of Newton were set up with an arbitrary choice of units of mass, length and time. These can always be altered at will

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*Further generalization is possible (Long, 1963; Birkhoff, 1950).

without changing the form of the equations, and the techniques of ordinary dimensional analysis in mechanics are simply reflections of this fact. But as we see in the above example, we do ourselves an injustice if we ignore other information about the mathematical nature of our laws.

3. Physical approximations and generalized dimensional analysis

Very frequently a generalized dimensional analysis such as that performed above in the example of the projectile, is no more informative than the ordinary analysis with its fundamental units of mass, length and time. Yet it also happens frequently that the combination of generalized dimensional analysis with the physical process of approximation is more informative. Suppose, for example, we are interested in the steady flow of an incompressible, viscous fluid over a half-infinite plate as

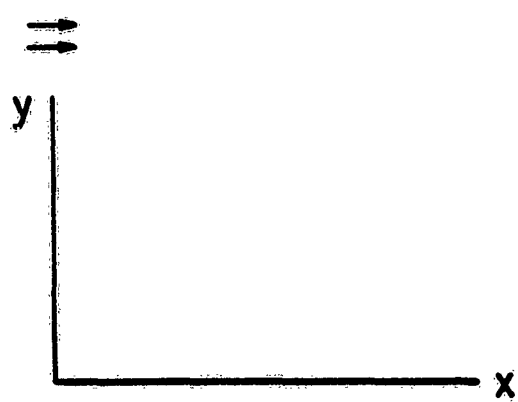

Fig. 2

shown in Figure 2. With the choice of axes as shown, the problem is probably determined by the following set of equations and conditions:

$$u u_x + v u_y = -\frac{1}{\rho} P_x + \nu (u_{xx} + u_{yy}) \tag{5}$$

$$u v_x + v v_y = -\frac{1}{\rho} P_y + \nu (v_{xx} + v_{yy}) \tag{6}$$

$$u_x + v_y = 0 \tag{7}$$

$$\text{At } x > 0, \quad y = 0, \quad u = v = 0, \quad \text{As } y \to \infty, \quad u \to U \tag{8}$$

Equation (7) shows that we may introduce a streamfunction ψ such that

$$u = -\psi_y \ , \quad v = \psi_x \tag{9}$$

The equations in (9) can be used to replace equation (7).

Obviously

$$\psi = f(x, y; \nu, U) \tag{10}$$

and ordinary dimensional analysis shows that ψ takes the form

$$\frac{\psi}{\sqrt{\nu U x}} = f\left(y \sqrt{\frac{U}{\nu x}}, \frac{y}{x}\right) \tag{11}$$

Furthermore, we may perform a generalized dimensional analysis of the system of equations (5), (6), (8) and (9) as in Section 2, and we find that we are led precisely to equation (11) again. But suppose that we allow ourselves to use the physical arguments of Prandtl (Goldstein, 1938) that when the viscosity is very small the influence of the plate is limited to a very thin layer in the immediate vicinity of the plate. It is then evident that changes in velocity in directions normal to the plate are very much greater than changes in the direction of the plate. To use this reasonable conjecture, we first cross-differentiate equations (5) and (6) to eliminate the pressure terms and we then introduce the stream-function. The governing equations and conditions are now

$$-\psi_y(\psi_{xxx}+\psi_{yyx})+\psi_x(\psi_{xxy}+\psi_{yyy}) = \nu(\psi_{xxxx}+2\psi_{xxyy}+\psi_{yyyy}) \tag{12}$$

$$\text{At } x>0,\ y=0,\ \psi_x=\psi_y=0,\ \text{As } y\to\infty,\ \psi_y\to -U \tag{13}$$

Now, with the assumption,

$$\frac{\partial^2}{\partial x^2} << \frac{\partial^2}{\partial y^2} \tag{14}$$

the governing equations and conditions take the form

$$-\psi_y\,\psi_{yyx}+\psi_x\,\psi_{yyy} = \nu\psi_{yyyy} \tag{15}$$

$$\text{At } x>0,\ y=0,\ \psi_x=\psi_y=0,\ \text{As } y\to\infty,\ \psi_y\to -U \tag{16}$$

We now perform a generalized dimensional analysis of this system, by assigning arbitrary dimensions to all quantities, requiring only that all terms in all equations have the same dimensions. Thus, let

$$[\psi] = A \qquad [\nu] = E$$
$$[x] = B \qquad [U] = F$$
$$[y] = C \tag{17}$$

tentatively. Then (15) and (16) require

$$\frac{A^2}{BC^3} = \frac{EA}{C^4} \qquad \frac{A}{C} = F$$

Therefore,

$$[\psi] = A \qquad [\nu] = \frac{AC}{B}$$
$$[x] = B \qquad [U] = \frac{A}{C}$$
$$[y] = C$$

We now have three fundamental units, A, B, C, instead of two, L and T, and if we now apply the π-theorem to (10) we get

$$\frac{\psi}{\sqrt{\nu U x}} = \phi\left(y\sqrt{\frac{U}{\nu x}}\right) \tag{18}$$

instead of the less informative (11). With the new independent variable,

$$\eta = y\sqrt{\frac{U}{\nu x}} \tag{19}$$

the partial differential equation (15) reduces to an <u>ordinary</u> differential equation in the single independent variable η . The problem can then be solved numerically.

4. Model of air flow over a mountain

To illustrate the application of the above discussions to the modeling problem, we consider the problem of modeling air flow over a mountain barrier. Using arguments (Long, 1959) which do not concern us here, we may show that for certain purposes a reasonably accurate governing differential equation is

$$\frac{\partial^2 y_0}{\partial x^2} + \frac{\partial^2 y_0}{\partial y^2} + \sigma^2 (y_0 - y) = 0 \tag{20}$$

where y_0 is the height far upstream (where the disturbance vanishes) of the streamline through the point (x, y), and σ^2 is a constant involving gravity, density stratification, and upstream velocity. It is also known that the motion is determined by the kinematic conditions at the "top" and bottom of the atmosphere and by the upstream conditions. Suppose, for simplicity, we take the model to be flow between two parallel, horizontal surfaces separated by a distance h, and over a rectangular obstacle of width b, height a at the bottom. Then we have

$$\begin{aligned} y_0 &= h \quad \text{At} \quad y = h \\ y_0 &= 0 \quad \text{At} \quad y = 0, \quad x \le -\frac{b}{2} \\ y_0 &= 0 \quad \text{At} \quad y = 0, \quad x \ge \frac{b}{2} \\ y_0 &= 0 \quad \text{At} \quad y = a, \quad -\frac{b}{2} \le x \le \frac{b}{2} \end{aligned} \tag{21}$$

The condition that the disturbance vanish upstream is

$$y_0 \to y \quad \text{As} \quad x \to \infty \tag{22}$$

Equations (20), (21) and (22) constitute a completely stated mathematical problem. If, however, it is too difficult to be solved, we may very well resort to experimentation (i.e., modeling). Since there is only one dimension, length, in this problem, ordinary dimensional analysis yields

$$\frac{y_o}{h} = f\left(\frac{y}{h}, \frac{x}{h}, \frac{a}{h}, \frac{b}{h}, \sigma^2 h^2\right) \tag{23}$$

and we find that we must have equal values of a/h, b/h, $\sigma^2 h^2$ if we are to have similar phenomena in any two experiments, or in model and prototype.

If we perform a generalized dimensional analysis of the problem, we find that we get the same result as in Eq. (23); but suppose conditions are such that the variations in the flow pattern are very gradual in the horizontal direction, compared to the vertical direction, i.e., suppose

$$\frac{\partial^2}{\partial x^2} << \frac{\partial^2}{\partial y^2}$$

Then we have conditions (21) and (22) but the governing differential equation is

$$\frac{\partial^2 y_o}{\partial y^2} + \sigma^2 (y_o - y) = 0$$

A generalized dimensional analysis now yields

$$\frac{y_o}{h} = f\left(\frac{y}{h}, \frac{x}{b}, \frac{a}{h}, \sigma^2 h^2\right)$$

and we find that we have similar phenomena by demanding equality of only two parameters, a/h and σ^2h^2, instead of three. The experimental advantages are very great of course, since much less experimentation is now necessary to explore the full range of "long-wave" patterns.